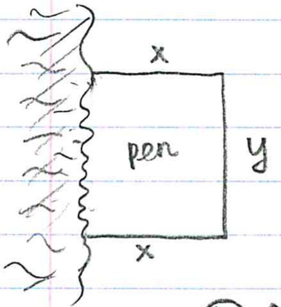
3.7 Optimization.

1. A farmer wants to build a rectangular pen, bounded on one side by a river and electric fence on the other sides. If he has 8m of wire to use, what is the largest area he can enclose?



ⓐ Area of pen: $A = xy$

Only in terms of x: $2x + y = 8 \Rightarrow y = 8 - 2x$

$$A(x) = x(8-2x) = 8x - 2x^2$$

ⓑ Critical pts.? $A'(x) = 8 - 4x \Rightarrow x = 2$ C.P.

ⓒ Max area? $A(2) = 2 \cdot 4 = 8$

Why is this the max (and not min, for example)?

FIRST DERIV. TEST FOR ABSOLUTE EXTREME VALUES

Suppose c is a critical number of a continuous function f on an interval.

ⓐ If $f'(x) > 0$, for all $x < c$ and $f'(x) < 0$, for all $x > c$, then $f(c)$ is the absolute maximum value of f.

ⓑ If $f'(x) < 0$, for all $x < c$ and $f'(x) > 0$, for all $x > c$, then $f(c)$ is the absolute minimum value of f.

ⓐ

x		c	
$f'(x)$	+ + + +	0	− − − −
$f(x)$	↗	max	↘

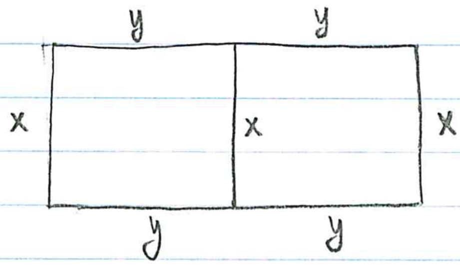
ⓑ

x		c	
$f'(x)$	− − −	0	+ + +
$f(x)$	↘	min	↗

$A(x)$ is defined here for $0 \le x \le 8$, but regardless:

x		2	
$A'(x)$	+ +	0	− − −
$A(x)$	↗	max 8	↘

② Rancher wants to fence in a rectangular area of $18\ m^2$ in a field then divide the region in half w/a fence down the middle parallel to one side. What is the <u>smallest</u> length of fencing required?



$$\text{Area} = x\cdot(2y) = 2xy = 18 \Rightarrow y = \frac{9}{x}$$

$$\text{Length} = 3x + 4y = 3x + \frac{36}{x} = \frac{3x^2+36}{x}$$

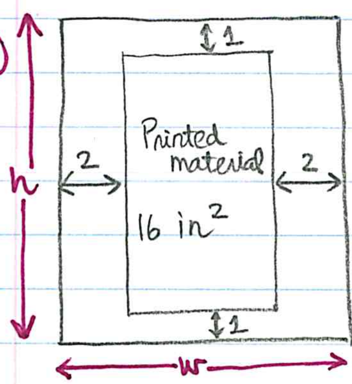
$$L(x) = 3x + \frac{36}{x} \Rightarrow L'(x) = 3 - \frac{36}{x^2} = \frac{3x^2-36}{x^2} = \frac{3(x^2-12)}{x^2}$$

x	$-\sqrt{12}$	0	$\sqrt{12}$
$L'(x)$	$+ \; + \; 0 \; -$	$\vert$	$- \; 0 \; + \; +$
$L(x)$	$\nearrow$ max $\searrow -\infty$	$\vert$	$+\infty \searrow L(\sqrt{12}) \nearrow$
			abs. min

$$L(\sqrt{12}) = 3\sqrt{12} + \frac{36}{\sqrt{12}}$$

abs. min.

③



Poster: top & bottom margins: 1 in

side margins: 2 in

Area of printed material = $16\ in^2$

w = width; h = height

ⓐ $A(w)$ = area of entire poster in terms of w only

16 = area of printed material

$16 = (w-4)(h-2)$

$$\Rightarrow h-2 = \frac{16}{w-4} \Rightarrow \boxed{h = \frac{16}{w-4} + 2}$$

$$\Rightarrow \boxed{A(w) = hw = \frac{16w}{w-4} + 2w}$$

ⓑ Dimensions of poster w/ <u>smallest area</u>? $(w>4,\ h>2)$

$$A'(w) = \frac{16(w-4) - 16w\cdot 1}{(w-4)^2} + 2 = \frac{16w - 64 - 16w}{(w-4)^2} + 2 = 2 - \frac{64}{(w-4)^2}$$

$$A'(w) = 0 \Rightarrow 2 = \frac{64}{(w-4)^2} \Rightarrow (w-4)^2 = 32 \Rightarrow w - 4 = \pm\sqrt{32}$$

$$\Rightarrow \boxed{w = 4 \pm 4\sqrt{2}}$$

(critical numbers) ↳

Of the critical numbers, only $4+4\sqrt{2}$ is in the domain $(w>4)$

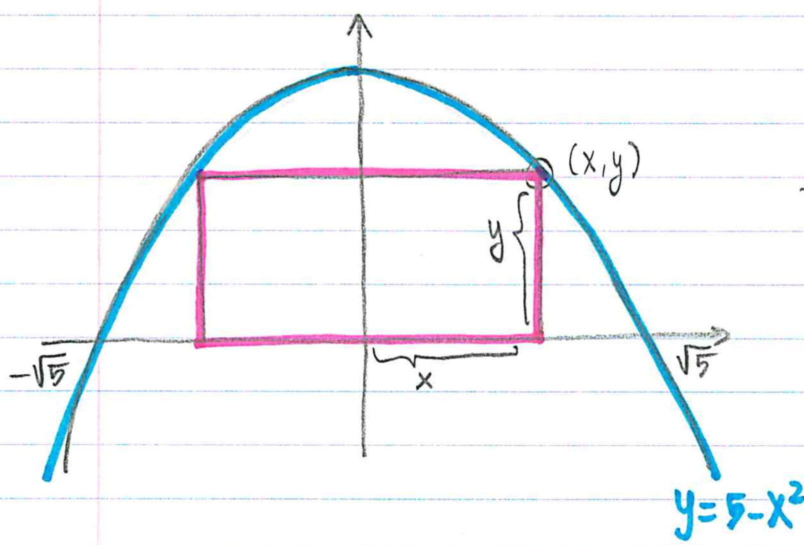
$\swarrow$ width when abs. min occurs

x	$4-4\sqrt{2}$ $\quad$ 4	$4+4\sqrt{2}$
$w'(x)$	$+\ +\ 0\ -$	$-\ -\ 0\ +\ +$
$w(x)$		$\searrow$ min $\nearrow$

$$A'(w) = 2 - \frac{64}{(w-4)^2} = \frac{2\left((w-4)^2 - 32\right)}{(w-4)^2}$$

when $\boxed{w = 4+4\sqrt{2}}$, $h = \dfrac{16}{4\sqrt{2}} + 2 \Rightarrow \boxed{h = 2\sqrt{2}+2}$

(4) A rectangle is inscribed w/ its base on the x-axis and its upper corners on the parabola $y = 5-x^2$. Dimensions of rectangle w/ greatest poss. area?



$$A = (2x)\cdot y$$
$$= (2x)\cdot(5-x^2)$$
$$= 10x - 2x^3$$

Domain: $x \in (-\sqrt{5}, \sqrt{5})$

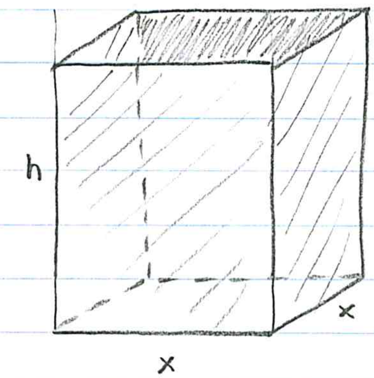
$$A'(x) = 10 - 6x^2$$
$$A'(x) = 0 \Rightarrow x^2 = \frac{10}{6} = \frac{5}{3}$$
$$\Rightarrow x = \pm\sqrt{\frac{5}{3}}$$

x	$-\sqrt{5}$	$-\sqrt{\frac{5}{3}}$	$\sqrt{\frac{5}{3}}$	$\sqrt{5}$
$A'(x)$	$-\ -$	0	$+\ +\ +\ 0$	$-\ -\ -$
$A(x)$	$\searrow$	min	$\nearrow$ max	$\searrow$

Max area occurs at $x = \sqrt{\frac{5}{3}}$, so $\boxed{w = 2\sqrt{\frac{5}{3}}}$ $\boxed{h = \frac{10}{3}}$

$$y = 5 - \frac{5}{3} = \frac{10}{3}$$

(5) A total of $24\ ft^2$ of material is to be used to make a box w/a square base and an open top. What is the largest possible volume?



Area of material: $4hx + x^2 = 24$

$\hookrightarrow 4hx = 24 - x^2$

$h = \dfrac{24 - x^2}{4x}$

Volume: $V = x^2 \cdot h$

$$V(x) = x^2 \cdot \frac{24 - x^2}{4x} = x \cdot \frac{24 - x^2}{4} = \frac{24x - x^3}{4}$$

$$V'(x) = \frac{1}{4}(24 - 3x^2)$$

$$V'(x) = 0 \Rightarrow 24 = 3x^2 \Rightarrow 8 = x^2 \Rightarrow x = \pm 2\sqrt{2}$$

(In this problem, $x > 0$)

x	$-2\sqrt{2}$		0	$2\sqrt{2}$	
$V'(x)$	$-$ 0	$+$ $+$		$+$ 0	$- - -$
$V(x)$	$\searrow$ min		$\nearrow$	max	$\searrow$

Max occurs at $x = \sqrt{8} = 2\sqrt{2} \Rightarrow V(2\sqrt{2}) = \dfrac{\overset{6}{\cancel{24}} \cdot 2\sqrt{2} - \overbrace{(2\sqrt{2})^3}^{16\sqrt{2}}}{\cancel{4}}$

$= 12\sqrt{2} - 4\sqrt{2} = 8\sqrt{2}$

(6) Point on the line $2x+y+3=0$ closest to the point $(-1,-4)$?

$d = \text{dist} = \sqrt{(x+1)^2+(y+4)^2}$ ← want to minimize
Same as minimizing d^2 (easier)

$D = d^2 = (x+1)^2+(y+4)^2$

$2x+y+3=0 \Rightarrow y=-2x-3$

$$D(x) = (x+1)^2 + (-2x-3+4)^2$$
$$= (x+1)^2 + (-2x+1)^2$$
$$= x^2+2x+1 + 4x^2-4x+1$$
$$= 5x^2-2x+2$$

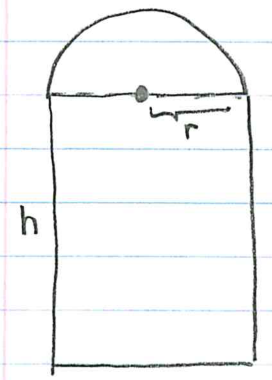
$D'(x) = 10x-2$

$D'(x)=0 \Rightarrow x = 1/5 \Rightarrow y = -\frac{2}{5}-3 \Rightarrow y = -17/5$

Domain for x, y in this problem is $(-\infty, \infty)$.

x	$1/5$
$D'(x)$	$- - -$ 0 $+ + + +$
$D(x)$	$\searrow$ min $\nearrow$

7 Norman window: Outside perimeter is 9m



ⓐ Area of window as a function of r only?

$$A = (\text{area of rectangle}) + (\text{area of semicircle})$$
$$= h(2r) + \tfrac{1}{2}\pi r^2$$

Perimeter $= 9 \Rightarrow 2h + 2r + \pi r = 9$

$$h = \frac{9-(\pi+2)r}{2}$$

$$\Rightarrow A = r\left(9-(\pi+2)r\right) + \tfrac{1}{2}\pi r^2$$
$$= 9r - (\pi+2)r^2 + \tfrac{1}{2}\pi r^2 = 9r - \left(\tfrac{1}{2}\pi + 2\right) r^2$$

ⓑ largest possible area?

$$A'(r) = 9 - 2\left(\tfrac{1}{2}\pi+2\right)r = 9 - (\pi+4)r$$

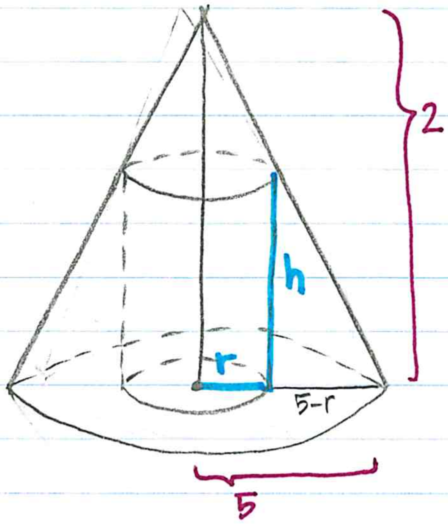
$A'(r) = 0$ when $r = \dfrac{9}{\pi+4}$

r	0	$\frac{9}{\pi+4}$	∞
$A'(r)$	$+ \ +$	0	$- \ - \ -$
$A(r)$	$\nearrow$	max	$\searrow$

Max area: $A\left(\dfrac{9}{\pi+4}\right) = \dfrac{81}{\pi+4} - \dfrac{\pi+4}{2}\cdot\left(\dfrac{9}{\pi+4}\right)^2$

$$= \frac{81}{\pi+4} - \frac{81}{2(\pi+4)} = \boxed{\frac{81}{2(\pi+4)}}$$

(8)



Cylinder inscribed in cone — height 2 in, base radius: 5 in

Dimensions of cylinder w/ max volume?

$$V = (\pi r^2) \cdot h$$

Relationship b/w r & h: (Similar triangles)

$$\frac{h}{2} = \frac{5-r}{5} \Rightarrow h = \frac{2(5-r)}{5}$$

$$V(r) = (\pi r^2) \cdot \frac{2(5-r)}{5} = \frac{2\pi(5r^2 - r^3)}{5}$$

$$V'(r) = \frac{2\pi}{5}(10r - 3r^2) = \frac{2\pi}{5} r(10-3r)$$

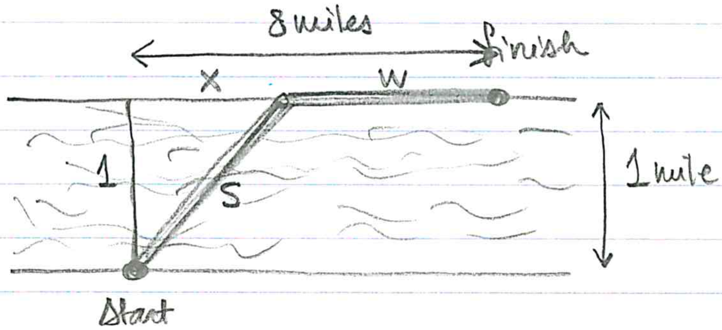
($r > 0$ in this problem) (actually $0 < r < 5$).

r		0		$10/3$		5	
$V'(r)$	$- -$	0	$+ \; +$	0	$-$		$-$
$V(r)$	$\searrow$	min	$\nearrow$	max	$\searrow$		

Dimensions: $r = \frac{10}{3}$ in

height: $h = \dfrac{2\left(5 - \frac{10}{3}\right)}{5} = \dfrac{2}{\cancel{5}} \cdot \dfrac{\cancel{5}}{3} = \dfrac{2}{3}$ in

(9) Woman standing @ edge of river (1 mile wide), wants to return to campsite on the other side of the river. She can — walk @ 5 mph, swim @ 3 mph.
First swim to cross river, then walk.
8 miles downstream from pt. directly across from her starting point.
What route takes least amt. of time?



S = distance she swims
W = distance she walks.

(a) How long to swim S miles? $\quad y \cdot 3 \frac{\text{miles}}{\text{hr}} = S \text{ miles} \Rightarrow y = \frac{S}{3} \text{ hr}$

(b) How long to walk w miles? $\quad z \cdot 5 \frac{\text{miles}}{\text{hr}} = W \text{ miles} \Rightarrow z = \frac{W}{5} \text{ hr}$

(c) Time to swim in terms of x?

$$x^2+1=S^2 \Rightarrow S=\sqrt{x^2+1} \ (\text{b/c } S>0) \Rightarrow y=\frac{\sqrt{x^2+1}}{3} \text{ hr}$$

(d) Time to walk in terms of x?

$$W = 8-x \Rightarrow z = \frac{8-x}{5} \text{ hr}$$

(e) T(x) = # of hours to swim & walk, $x \in (0,8)$

$$T(x) = \frac{1}{3}\sqrt{x^2+1} + \frac{1}{5}(8-x)$$

(f) Critical numbers?

$$T'(x) = \frac{1}{3}\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} - \frac{1}{5} = \frac{x}{3\sqrt{x^2+1}} - \frac{1}{5} = \frac{5x-3\sqrt{x^2+1}}{15\sqrt{x^2+1}}$$

$$5x = 3\sqrt{x^2+1}\ ; \quad 25x^2 = 9x^2+9; \quad 16x^2=9; \quad x^2=\frac{9}{16} \Rightarrow x=\frac{3}{4}$$

(g) $$T(3/4) = \frac{1}{3}\sqrt{\frac{9}{16}+1} + \frac{1}{5}\left(8-\frac{3}{4}\right) = \frac{1}{3}\cdot\frac{5}{4} + \frac{1}{5}\cdot\frac{29}{4} = \frac{5}{12}+\frac{29}{20}$$